Vasicek Bond Price Under The Euler Discretization

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The Vasicek model is a mathematical model that describes the evolution of interest rates. Vasicek models the short rate as a Ornstein-Uhlenbeck process. The short rate is the annualized interest rate at which an entity can borrow money for an infinitesimally short period of time. An Ornstein-Uhlenbeck process is a mean-reverting process where the short rate is allowed to incorporate random shocks but is pulled back to its long-term mean whenever it moves away from it. Interest rates exhibit mean reversion, which is the tendency for a stochastic process to return over time to a long-term mean. Vasicek's stochastic differential equation that describes the evolution of the short rate r_t in continuous time is...

$$\delta r_t = a(b - r_t)\,\delta t + \sigma\,\delta W_t \tag{1}$$

When the short rate moves below its long-term mean b the short rate drift becomes positive and the short rate is pulled upward. When the short rate moves above its long-term mean the short rate drift becomes negative and the short rate is pulled downward. The speed at which the drift is pulled upward of downward is given by the positive valued parameter a, which measures the speed of mean reversion. The greater the speed the faster the process reverts toward the long-term mean. Random shocks are introduced via the variables σ, which is annualized short rate volatility, and W_t, which is a Brownian motion with mean zero and variance t.

The short rate in time period t is r_0, which is the short rate in period zero (known), plus the sum of the changes in the short rate from period zero to period t (unknown). The equation for the short rate in continuous time is...

$$r_t = r_0 + \int\limits_0^t \delta r_s \tag{2}$$

The price of a pure discount bond may be written as the expectation of the path integral of the short-term interest rate. The price at time zero of a pure discount bond that matures at time T and pays one dollar at maturity can be written as...

$$B(0,T) = \mathbb{E}^{\hat{P}}\left[e^{-\int\limits_0^T r_s \delta s}\,\middle|\, I_0\right] \tag{3}$$

The variable r_s denotes the instantaneous short rate at time period $0 \leq s \leq T$ and I_0 is the information revealed up to time zero. The expectation is taken under the risk-neutral measure $\hat{P}$. Note that the solution to the integral in equation (3) above is the average interest rate for the period zero to period T. This average interest rate is referred to as the stochastic discount rate.

A Bond Pricing Problem

Assume that we want to price a default-free pure discount bond that pays $1,000 at maturity and matures in 3 years. The I_0 in equation (3) above, information revealed up to time zero, are the parameters to the problem.

Parameters to the problem:

T	=	Time to maturity	=	3 years
r_0	=	Current short rate	=	6.00%
b	=	Long-term short rate	=	10.00%
a	=	Mean reversion rate	=	0.40
σ	=	Short rate volatility	=	4.00%

Steps to the solution:

We can use bond price equation (3) as a template to determine the processes for which we need models and/or equations. The steps to pricing this bond are...

Step 1	-	We need a model for the short rate process r_s
Step 2	-	We need to evaluate the integral $\int_0^T r_s \delta s$
Step 3	-	We need to evaluate the expectation $\mathbb{E}[...]$

Step 1 - A Model For The Short Rate Process

We can use a Euler sheme to simulate the first-order discrete approximation to the continuous time short rate process described in equation (1) above. The idea is to approximate the stochastic differential equation for the short rate by its first order Taylor approximation. We will divide the time interval $[0, T]$ into k equal intervals of length h such that $T = kh$. If we set the time interval length equal to one month then the model parameters h and k are...

h	=	0.0833 (1/12th of one year (i.e. one month))
k	=	36 (such that T = 3 years = kh)

Vasicek's continuous time stochastic differential equation in discrete time becomes...

$$\begin{aligned} \Delta r_j &= a(b - r_j)h + \sigma\,(W_{j+1} - W_j) \\ &= a(b - r_j)h + \sigma\sqrt{h}\,z_{j+1} \end{aligned} \tag{4}$$

Note that the Brownian motion W_t in equation (1) above, which has mean zero and variance t, is replaced by a process where the zs are independent standard normal variates. The random variate z has mean zero and variance one, which is why we scale these variates by the square root of h.

We will define $\theta = (1 - ah)$. The equation for the short rate in any time period t as a function of the short rate in the prior period and a random shock is...

$$\begin{aligned} r_{t+1} &= r_t + \Delta r_t \\ &= r_t + a(b - r_t)h + \sigma\sqrt{h}\,z_{t+1} \\ &= r_t + abh - r_t ah + \sigma\sqrt{h}\,z_{t+1} \\ &= r_t(1 - ah) + abh + \sigma\sqrt{h}\,z_{t+1} \\ &= r_t\theta + abh + \sigma\sqrt{h}\,z_{t+1} \end{aligned} \tag{5}$$

The short rate in period zero can be observed at $t = 0$ and is therefore known. The equation for the short rate in period one as a function of the short rate in period zero and the random variate z_1 is...

$$\begin{aligned} r_1 &= r_0 + \Delta r_0 \\ &= r_0\theta + abh + \sigma\sqrt{h}\,z_1 \end{aligned} \tag{6}$$

The equation for the short rate in period two as a function of the short rate in period zero and the random variates z_1 and z_2 is...

$$\begin{aligned} r_2 &= r_1 + \Delta r_1 \\ &= r_1\theta + abh + \sigma\sqrt{h}\,z_2 \\ &= [r_0\theta + abh + \sigma\sqrt{h}\,z_1]\theta + abh + \sigma\sqrt{h}\,z_2 \\ &= r_0\theta^2 + abh + abh\theta + \sigma\sqrt{h}\,z_2 + \sigma\sqrt{h}\,z_1\theta \end{aligned} \tag{7}$$

The equation for the short rate in period three as a function of the short rate in period zero and the random variates z_1, z_2 and z_3 is...

$$\begin{aligned}
r_3 &= r_2 + \Delta r_2 \\
&= r_2\theta + abh + \sigma\sqrt{h}\, z_3 \\
&= [r_0\theta^2 + abh + abh\theta + \sigma\sqrt{h}\, z_2 + \sigma\sqrt{h}\, z_1\theta]\theta + abh + \sigma\sqrt{h}\, z_3 \\
&= r_0\theta^3 + abh + abh\theta + abh\theta^2 + \sigma\sqrt{h}\, z_3 + \sigma\sqrt{h}\, z_2\theta + \sigma\sqrt{h}\, z_1\theta^2
\end{aligned} \tag{8}$$

Note the pattern that is evolving. The equation for the short rate in any period t as a function of the short rate in period zero and t random variates is...

$$r_t = r_0\theta^t + abh\sum_{i=0}^{t-1}\theta^i + \sigma\sqrt{h}\sum_{i=0}^{t-1}\theta^i z_{t-i} \tag{9}$$

The mean of short rate equation (9) is...

$$\begin{aligned}
\mathbb{E}\Bigg[r_t\Bigg] &= \mathbb{E}\Bigg[r_0\theta^t + abh\sum_{i=0}^{t-1}\theta^i + \sigma\sqrt{h}\sum_{i=0}^{t-1}\theta^i z_{t-i}\Bigg] \\
&= \mathbb{E}\Bigg[r_0\theta^t\Bigg] + \mathbb{E}\Bigg[abh\sum_{i=0}^{t-1}\theta^i\Bigg] + \mathbb{E}\Bigg[\sigma\sqrt{h}\sum_{i=0}^{t-1}\theta^i z_{t-i}\Bigg] \\
&= r_0\theta^t + abh\frac{1-\theta^t}{ah} + 0 \\
&= r_0\theta^t + b(1-\theta^t) \\
&= b + (r_0 - b)\theta^t
\end{aligned} \tag{10}$$

The variance of short rate equation (9) is...

$$\begin{aligned}
\mathbb{E}\Bigg[r_t^2\Bigg] - \mathbb{E}\Bigg[r_t\Bigg]^2 &= \mathbb{E}\Bigg[\Bigg\{\sigma\sqrt{h}\sum_{i=0}^{t-1}\theta^i z_{t-i}\Bigg\}^2\Bigg] \\
&= \mathbb{E}\Bigg[\sigma\sqrt{h}\sum_{m=0}^{t-1}\theta^m z_{t-m} \times \sigma\sqrt{h}\sum_{n=0}^{t-1}\theta^n z_{t-n}\Bigg] \\
&= \sigma^2 h\,\mathbb{E}\Bigg[\sum_{m=0}^{t-1}\theta^m z_{t-m} \times \sum_{n=0}^{t-1}\theta^n z_{t-n}\Bigg] \\
&= \sigma^2 h\sum_{i=0}^{t-1}\theta^{2i} \\
&= \sigma^2 h\frac{1-\theta^{2t}}{1-\theta^2}
\end{aligned} \tag{11}$$

Note: The covariance between z_m and z_n is one when $m = n$ and zero when $m \neq n$. When multiplying vectors we can eliminate any instances where $m \neq n$ from our computation. We can do this because the zs are independent.

Step 2 - Evaluate The Integral For Stochastic Discount Rate

We will define $\hat{r}$ to be the stochastic discount rate for our bond pricing problem. The stochastic discount rate is the integral within equation (3) above. If we define T as time to maturity, the equation for the stochastic discount rate in continuous time is...

$$\hat{r} = \int_0^T r_s \delta s \tag{12}$$

The equation for stochastic discount rate in discrete time where $\hat{r}$ is the average interest rate outstanding is...

$$
\begin{aligned}
\hat{r} &= \frac{1}{2}r_0 h + \sum_{j=1}^{k-1} r_j\, h + \frac{1}{2}r_k h \\
&= \frac{1}{2}r_0 h + \sum_{j=1}^{k-1}\left[r_0\theta^j + abh\sum_{i=0}^{j-1}\theta^i + \sigma\sqrt{h}\sum_{i=0}^{j-1}\theta^i z_{k-i}\right] h + \frac{1}{2}r_k h \\
&= \frac{1}{2}r_0 h + \sum_{j=1}^{k-1} r_0\theta^j h + \sum_{j=1}^{k-1} abh\sum_{i=0}^{j-1}\theta^i h + \sum_{j=1}^{k-1}\sigma\sqrt{h}\sum_{i=0}^{j-1}\theta^j z_{k-i}\, h + \frac{1}{2}r_k h \\
&= \frac{1}{2}r_0 h + \frac{r_0\theta}{a}\left(1-\theta^{k-1}\right) + b(k-1)h - \frac{b\theta}{a}(1-\theta^{k-1}) + \sigma h^{\frac{3}{2}}\sum_{j=1}^{k-1}\sum_{i=0}^{j-1}\theta^i z_{j-i} + \frac{1}{2}r_0\theta^k h \\
&+ \frac{1}{2}bh(1-\theta^k) + \frac{1}{2}\sigma h^{\frac{3}{2}}\sum_{i=0}^{k-1}\theta^i z_{k-i} \\
&= r_0 h\left[\frac{1}{2} + \frac{\theta(1-\theta^{k-1})}{ah} + \frac{1}{2}\theta^k\right] + bh\left[(k-1) - \frac{\theta(1-\theta^{k-1})}{ah} + \frac{1}{2}(1-\theta^k)\right] + \sigma h^{\frac{3}{2}}\frac{1}{ah}\sum_{i=0}^{k-1}(1-\Phi\theta^i)z_{k-i} \qquad (13)
\end{aligned}
$$

The stochastic discount rate mean is...

$$
\mathbb{E}\left[\hat{r}\right] = r_0 h\left[\frac{1}{2} + \frac{\theta(1-\theta^{k-1})}{ah} + \frac{1}{2}\theta^k\right] + bh\left[(k-1) - \frac{\theta(1-\theta^{k-1})}{ah} + \frac{1}{2}(1-\theta^k)\right] \qquad (14)
$$

The stochastic discount rate variance is...

$$
\mathbb{E}\left[\hat{r}^2\right] - \mathbb{E}\left[\hat{r}\right]^2 = \sigma^2 h^3\frac{1}{(ah)^2}\left[k - 2\Phi\frac{1-\theta^k}{ah} + \Phi^2\frac{1-\theta^{2k}}{1-\theta^2}\right] \qquad (15)
$$

Note: We utilize equations in the appendix for the derivations above.

Step 3 - Evaluate the Expectation

We will define F as the dollar amount paid when a pure discount bond matures at time T. We will model the short rate process per equation (13) above. To generate a single interest rate path using this approximation we draw k independent normal variates. We will define $f(h,n)$ as the price of this bond given a time interval of length h and short rate path number n (the number of possible interest rate paths is infinite). The equation for bond price given these parameters and one interest rate path is...

$$
f(h,n) = F \times exp\left[-\left\{\frac{1}{2}r_0 + \sum_{j=1}^{k-1} r_j + \frac{1}{2}r_k\right\}h\right] \qquad (16)
$$

We can estimate bond price by running equation (16) through a Monte Carlo simulation with N number of trials. The price of the bond would equal the mean of simulation results. We can rewrite Vasicek's continuous time bond price equation (3) in discrete time as...

$$
\hat{B}(0,T) = \frac{1}{N}\sum_{n=1}^{N} -f(h,n) \qquad (17)
$$

Problem Solution and Sanity Check

Bond price via Monte Carlo simulation equation (17) and 100,000 trials = 797.34

Sanity check:
Stochastic discount rate mean via equation (14) = 0.2307
Stochastic discount rate variance via equatioh (15) = 0.0066
Bond price = 1,000 x exp(-0.2307 + 0.0066/2) = 796.60

Postscript

We often need to know the expected number of periods that transpire until the current short rate mean increases or decreases to a given future short rate mean. Starting with equation (10) above, the expected number of periods (t) for the current short rate r_0 to reach r_n is...

$$
\begin{aligned}
r_t &= b + (r_0 - b)\theta^t \\
\theta^t &= \frac{r_t - b}{r_0 - b} \\
t \ln \Bigg[\theta\Bigg] &= \ln \Bigg[\frac{r_t - b}{r_0 - b}\Bigg] \\
t &= \ln \Bigg[\frac{r_t - b}{r_0 - b}\Bigg] \div \ln \Bigg[\theta\Bigg] \qquad (18)
\end{aligned}
$$

Appendix

We can rewrite the equation $\sum\limits_{j=1}^{k-1} r_0 \theta^j h$, which is used in equation (13) above, as...

$$
\begin{aligned}
\sum_{j=1}^{k-1} r_0 \theta^j h &= r_0 h \sum_{j=1}^{k-1} \theta^j \\
&= r_0 h \Bigg[\Bigg\{\sum_{j=0}^{k-1} \theta^j\Bigg\} - 1\Bigg] \\
&= r_0 h \Bigg[\frac{1 - \theta^k}{ah} - 1\Bigg] \\
&= r_0 h \Bigg[\frac{\theta - \theta^k}{ah}\Bigg] \\
&= \frac{r_0 \theta}{a}\,(1 - \theta^{k-1}) \qquad (19)
\end{aligned}
$$

We can rewrite the equation $\sum\limits_{j=1}^{k-1} abh \sum\limits_{i=0}^{j-1} \theta^i h$, which is used in equation (13) above, as...

$$
\begin{aligned}
\sum_{j=1}^{k-1} abh \sum_{i=0}^{j-1} \theta^i h &= abh^2 \sum_{j=1}^{k-1} \sum_{i=0}^{j-1} \theta^i \\
&= abh^2 \sum_{j=1}^{k-1} \frac{1 - \theta^j}{ah} \\
&= bh \sum_{j=1}^{k-1} (1 - \theta^j) \\
&= bh \Bigg[(k-1) - \Bigg\{\Bigg[\sum_{j=0}^{k-1} \theta^j\Bigg] - 1\Bigg\}\Bigg] \\
&= bh \Bigg[(k-1) - \Bigg\{\frac{1 - \theta^k}{ah} - 1\Bigg\}\Bigg] \\
&= bh \Bigg[(k-1) - \frac{\theta - \theta^k}{ah}\Bigg] \\
&= b(k-1)h - \frac{b\theta}{a}(1 - \theta^{k-1}) \qquad (20)
\end{aligned}
$$

We can rewrite the equation $\sum\limits_{j=1}^{k-1} \sigma\sqrt{h} \sum\limits_{i=0}^{j-1} \theta^j z_{j-i}\, h$, which is used in equation (13) above, as...

$$\sum_{j=1}^{k-1} \sigma\sqrt{h} \sum_{i=0}^{j-1} \theta^j z_{j-i}\, h = \sigma h^{\frac{3}{2}} \sum_{j=1}^{k-1} \sum_{i=0}^{j-1} \theta^j z_{j-i} \tag{21}$$

We can rewrite the equation $\frac{1}{2} r_t h$, which is used in equation (13) above, as...

$$\begin{aligned}
\frac{1}{2} r_k h &= \frac{1}{2}\left[r_0 \theta^k + abh \sum_{i=0}^{k-1} \theta^i + \sigma\sqrt{h} \sum_{i=0}^{k-1} \theta^i z_{k-i} \right] h \\
&= \frac{1}{2}\left[r_0 \theta^k + abh \frac{1-\theta^k}{ah} + \sigma\sqrt{h} \sum_{i=0}^{k-1} \theta^i z_{k-i} \right] h \\
&= \frac{1}{2} r_0 \theta^k h + \frac{1}{2} bh(1-\theta^k) + \sigma h^{\frac{3}{2}} \sum_{i=0}^{k-1} \theta^i z_{k-i}
\end{aligned} \tag{22}$$

We can rewrite the equation $\sum\limits_{j=1}^{k-1} \sum\limits_{i=0}^{j-1} \theta^i z_{j-i} + \frac{1}{2} \sum\limits_{i=0}^{k-1} \theta^i z_{k-i}$, which is used in equation (13) above, as...

First step - Develop the pattern

$$\begin{array}{llllll}
\text{k=1} & : & [\theta^0 + \theta^1 + \theta^2 + \frac{1}{2}\theta^3] z_1 & = & [\frac{1-\theta^3}{ah} + \frac{1}{2}\theta^3] z_1 & = & [1 - \theta^3(1 - \frac{1}{2}ah)] z_1 \\
\text{k=2} & : & [\theta^0 + \theta^1 + \frac{1}{2}\theta^2] z_2 & = & [\frac{1-\theta^2}{ah} + \frac{1}{2}\theta^2] z_2 & = & [1 - \theta^2(1 - \frac{1}{2}ah)] z_2 \\
\text{k=3} & : & [\theta^0 + \frac{1}{2}\theta^1] z_3 & = & [\frac{1-\theta^1}{ah} + \frac{1}{2}\theta^1] z_3 & = & [1 - \theta^1(1 - \frac{1}{2}ah)] z_3 \\
\text{k=4} & : & [\frac{1}{2}\theta^0] z_4 & = & [\frac{1-\theta^0}{ah} + \frac{1}{2}\theta^0] z_4 & = & [1 - \theta^0(1 - \frac{1}{2}ah)] z_4
\end{array}$$

Second step - Define a new variable Φ

$$\Phi = 1 - \frac{ah}{2}$$

Third step - Rewrite the equation

$$\sum_{j=1}^{k-1} \sum_{i=0}^{j-1} \theta^i z_{j-i} + \frac{1}{2} \sum_{i=0}^{k-1} \theta^i z_{k-i} = \frac{1}{ah} \sum_{i=0}^{k-1} (1 - \Phi\theta^i) z_{k-i} \tag{23}$$

We can rewrite the square of equation (23) above, which is the stochastic discount rate variance, as...

$$\begin{aligned}
\mathbb{E}\left[\left\{ \frac{1}{ah} \sum_{i=0}^{k-1} (1 - \Phi\theta^i) z_{k-i} \right\}^2 \right] &= \mathbb{E}\left[\frac{1}{(ah)^2} \sum_{i=0}^{k-1} (1 - \Phi\theta^i) z_{k-i} \sum_{j=0}^{k-1} (1 - \Phi\theta^j) z_{k-j} \right] \\
&= \frac{1}{(ah)^2} \sum_{i=0}^{k-1} (1 - \Phi\theta^i)^2 \\
&= \frac{1}{(ah)^2} \sum_{i=0}^{k-1} (1 - 2\Phi\theta^i + \Phi^2\theta^{2i}) \\
&= \frac{1}{(ah)^2} \left[\sum_{i=0}^{k-1} 1 - 2\Phi \sum_{i=0}^{t-1} \theta^i + \Phi^2 \sum_{i=0}^{k-1} \theta^{2i} \right] \\
&= \frac{1}{(ah)^2} \left[k - 2\Phi \frac{1-\theta^k}{ah} + \Phi^2 \frac{1-\theta^{2k}}{1-\theta^2} \right]
\end{aligned} \tag{24}$$